waves in a stratified layer it depends on Fr: as $Fr \to 0$ it approaches zero, while as $Fr \to \infty$ it approaches θ_W.

An even smaller difference is observed in the behavior of waves at the boundary of the wedge itself. The crests at the boundary of the wedge of surface ship waves make an angle of $\varphi_W = \arctan\sqrt{2}$ with the x axis. In the case of internal ship waves as $Fr \to \infty$ it approaches φ_W, and as Fr decreases it slowly grows and approaches 60°. Figure 6 shows the theoretical dependence of the slope angle of the tangent at the turning point of the line of constant phase with respect to the x axis on 1/Fr.

For small values of Fr the experimentally observed lines of constant phase are practically parallel straight lines. In addition, they pass outside the boundary of the wave zone, determined by the value of θ ($\theta \to 0$).

We thank A. T. Onufriev for a discussion of the results obtained.

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SOME CLASSES OF TWO-DIMENSIONAL VORTEX FLOWS OF AN IDEAL FLUID

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UDC 532.5+517.958

There are relatively few known exact steady solutions of the two-dimensional Euler equations [1-3]. This is in part explained by the fact that the symmetry group of these equations is low [4]. But progress achieved in the study of nonlinear wave equations [5, 6] can be partially carried over to the study of elliptic problems. The purpose of the present paper is to obtain solutions of the equation for the stream function and to analyze these solutions. The solutions found here describe motion of the source type in a rotating fluid, periodic flow between two walls, motion in a rectangular cylinder, and others.

1. The stream function ψ for the two-dimensional steady flow of an ideal fluid satisfies the equation

$$\Delta\psi(x, y) = \omega, \tag{1.1}$$

where the vorticity ω is a function of ψ. For certain forms of the right hand side (1.1) can be solved using a modified separation of variables method and the Beklund transformation [6].

We assume that the vorticity is given by $\omega(\psi) = \varepsilon \sin \psi$ ($\varepsilon = \pm 1$). We look for a solution of (1.1) in the form [5] $\psi(x, y) = 4 \arctan(f(x)g(y))$, where the functions f and g satisfy the ordinary differential equations

$$f'^2 = nf^4 + mf^2 + k, \quad g'^2 = kg^4 + (\varepsilon - m)g^2 + n \tag{1.2}$$

Krasnoyarsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 109-117, January-February, 1989. Original article submitted October 6, 1987.

(n, m, k are arbitrary constants). Then the vorticity and the components of the velocity vector can be calculated from the relations

$$\omega = 4\varepsilon \frac{fg(1-f^2g^2)}{(1+f^2g^2)^2}; \tag{1.3}$$

$$u = 4\frac{fg'}{1+f^2g^2}, \quad v = -4\frac{f'g}{1+f^2g^2}. \tag{1.4}$$

The choice of the constants in (1.2) determines the type of flow. Suppose n > 0, k, m < 0, and $\varepsilon = -1$. We consider three cases: a) $4kn < m^2 < 1/4$; b) $4kn = m^2 < 1/4$; c) $4kn = m^2 = 1/4$. We first discuss the most degenerate situation, case c. In this case (1.2) is satisfied by the functions $f(x) = \frac{1}{2\sqrt{n}}\tanh\left(\frac{x}{2}\right)$, $g(y) = 2\sqrt{n}\tanh\left(\frac{y}{2}\right)$. Then in the plane of flow $R^2(x, y)$ the straight lines x = 0 and y = 0 are streamlines and the vorticity is zero on these lines. It follows from (1.3) that $\omega < 0$ in the first and third quadrants of the plane $R^2(x, y)$, while $\omega > 0$ in the second and fourth quadrants. Since the trajectories are determined by the relation $\tanh\left(\frac{x}{2}\right)\tanh\left(\frac{y}{2}\right) = c$, where $|c| < 1$, each stream line has its own pair of asymptotes x = c and y = c. Because the functions f and g are odd, it is sufficient to know the streamlines in the first quadrant of the plane $R^2(x, y)$. The equation of the trajectories solved for y has the form $y(x) = 2\,\mathrm{arth}\left(c\coth\left(\frac{x}{2}\right)\right)$ $(0 < c < 1)$. Therefore the function y(x) monotonically decreases for x > 2 arth c. The components of the velocity vector are

$$u = \frac{2\tanh\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{y}{2}\right)\left(1+\tanh^2\left(\frac{x}{2}\right)\tanh^2\left(\frac{y}{2}\right)\right)}, \quad v = \frac{-2\tanh\left(\frac{y}{2}\right)}{\cosh^2\left(\frac{x}{2}\right)\left(1+\tanh^2\left(\frac{x}{2}\right)\tanh^2\left(\frac{y}{2}\right)\right)},$$

and hence $u \to 0$ in the limit $|y| \to \infty$ and $v \to 0$ in the limit $|x| \to \infty$. In addition, the vorticity goes to zero when x and y both go to infinity. This solution can be interpreted as flow inside a right angle bounded by the coordinate axes, or as the symmetric collision of two diffuse "jets." The "jets" are diffuse since the pressure is not constant along any of the streamlines.

Suppose we have condition b. Then by a scale transformation equation (1.2) for g reduces to the standard form $g'^2 = (1 - g^2) \times (1 - p^2g^2)$, where the constant p can be expressed in terms of n, m, and k. Hence the function g can be expressed in terms of the elliptic sine function sin y [7, 8] with a certain period T and two zeroes per period. In view of the translational invariance of (1.2), we can put g(0) = 0. The function f is given by $f = \sqrt{\frac{-m}{2n}}\tanh\left(\sqrt{\frac{-m}{2}}x\right)$. Therefore the straight lines x = 0, y = TL/2 are streamlines (L is any integer). In view of the periodicity of g and the fact that f and g are odd functions, it is sufficient to know the streamlines inside the half-strip $\{0 \leqslant y \leqslant T/2,\ x \geqslant 0\}$. Since $\lim_{x\to\infty}\tanh\left(\sqrt{\frac{-m}{2}}x\right) = 1$, any trajectory $\tanh\left(\sqrt{\frac{-m}{2}}x\right)g(y) = c$, where $c \neq 0$, has a pair of asymptotes parallel to the x axis. The equation for the trajectories determines the implicit function x(y), whose derivative is $\frac{dx}{dy} = -c\sqrt{\frac{2}{-m}}\frac{g'}{g^2}\cosh^2\left(\sqrt{\frac{-m}{2}}x\right)$, and therefore it is not difficult to obtain a qualitative picture of the streamlines (Fig. 1). We note that the velocity is finite but nonzero in the limit $x \to \infty$.

Finally in case *a* each of the equations of (1.2) can be transformed to the form $h'^2 = (1 - h^2)(1 - p^2h^2)$ $(p \in R)$ by means of a scale transformation. It follows at once that f and g can be expressed in terms of elliptic sine functions [7, 8] of periods t_1 and t_2 and both functions have two zeroes per period. With no loss of generality we can assume that f(0) = g(0) = 0. Therefore the straight lines $x = t_1L/2$ and $y = t_2L/2$ (L is any integer) are streamlines. In view of the periodicity of f and g and the fact that they are odd functions, the streamlines can be obtained over the entire plane of flow knowing only the behavior of the trajectories inside a rectangular cell $P = \{0 \leqslant x \leqslant t_1/2,\ 0 \leqslant y \leqslant t_2/2\}$. All streamlines belonging to P are closed and bounded. Indeed, the derivatives of f and g are

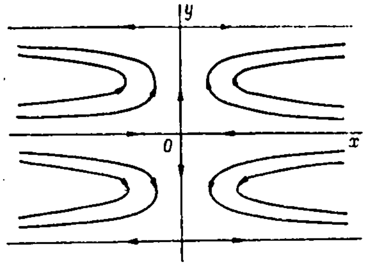


Fig. 1

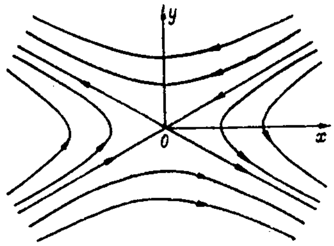


Fig. 2

nonzero except at the center of P, while the functions themselves do not have zeroes inside P and reach extremum values at the center of the cell. The streamlines are isolines of the function $F(x, y) = f(x)g(y)$. The function F reaches an extremum at the center of P, and its gradient is nonzero inside P. This means, according to the regular interval theorem [9], that the isolines inside P are closed and bounded. Therefore the plane of flow splits up into rectangular cells of the type P. There is a vortex inside each cell. Equating the left hand sides of (1.2) to zero, we can easily find the extremum values of the functions f and g and show that $|fg| < 1$. Then it can be shown with the help of (1.3) that the vorticity goes to zero on the boundary of the cell and changes sign when passing through a side of the rectangle P. The vortices in neighboring cells also rotate in different directions. Since the vorticity as a function of fg is $\omega(c) = -4c(1 - c^2)/(1 + c^2)^2$, then depending on whether the absolute value of the product fg reaches the value $\sqrt{3 - \sqrt{8}}$ (ω has an extremum for this value) or not, the vorticity will be an extremum either on the line $|fg| = \sqrt{3 - \sqrt{8}}$ or in the center of the cell. If we assume the boundary of P is a solid wall, we have flow inside a rectangular cylinder.

A different solution of (1.2) is obtained if we choose the parameters $\varepsilon = 1$, $n = 0$, $k < 0$, $0 < m < 1$. In this case the product $fg = \sqrt{\frac{1-m}{m}}\frac{\cosh(\sqrt{m}\,x)}{\cosh(\sqrt{1-m}\,y)}$. Hence the straight lines $y = \pm\sqrt{\frac{m}{1-m}}\,x$ are streamlines. Any other trajectory has a pair of asymptotes parallel to these lines. The vorticity changes sign upon passing through these straight lines. The velocity and vorticity go to zero in the limit $x \to \infty$ along any of the family of straight lines $y = ax$ $\left(a \neq \pm\sqrt{\frac{m}{1-m}}\right)$. This solution can be interpreted either as flow inside a two-sided angle (but not necessarily a right angle, as in case c) or as the collision of two diffuse "jets." A detailed discussion is omitted since this solution is similar to those discussed above. The qualitative picture of the streamlines is shown in Fig. 2.

Solutions of the source or sink type in a rotating fluid are of interest. One of the simplest solutions of (1.2) is $f(x) = \sqrt{k}\,x$, $g(y) = \frac{1}{\sqrt{k}\sinh y}$, corresponding to the parameters $\varepsilon = 1$, $m = n = 0$, $k > 0$. According to (1.4), the components of the velocity vector are $u = \frac{-4x\cosh y}{x^2 + \sinh^2 y}$, $v = \frac{-4\sinh^2 y}{x^2 + \sinh^2 y}$, and the vorticity is given by $\omega = 4\frac{x\sinh y(\sinh^2 y - x^2)}{(\sinh^2 y + x^2)^2}$. The streamlines are given by the equation $x = c\sinh y$ ($c \in R$). The origin of the coordinate system is a point of discontinuity for the velocity and the vorticity. However the flux $Q = \oint v dx + u dy$ is finite in this case and equal to -8π. (It is convenient to evaluate the integral along the contour $x^2 + \sinh^2 y = 1$). Hence there is a sink at the origin. The vorticity is zero on the lines $x = \pm\sinh y$, $x = 0$, $y = 0$, except for the point of discontinuity, and it changes sign upon passing through these lines. The velocity goes to zero in the limit $x \to \infty$ and fixed y. If we take any streamline $x = \sinh y$ and follow the change in velocity along it, then $u \to -4c/(1 + c^2)$, $v \to 0$, when x and y go to infinity along the trajectory and the velocity will reach a maximum when $|c| = 1$. In contrast to the potential solution for a sink, which is centrally symmetric, the flow found here resembles a four-current type of motion.

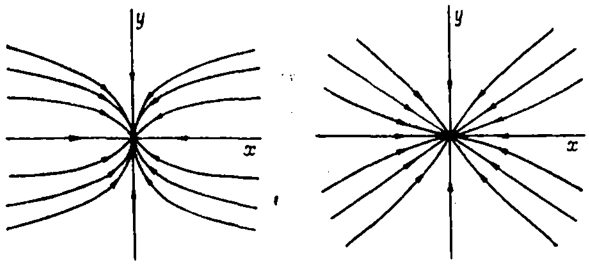


Fig. 3

The above solution is the limiting case when m → 0 of the solution $f = \sqrt{\frac{k}{m}}\sinh(\sqrt{m}\,x)$, $g = \sqrt{\frac{1-m}{k}}\frac{1}{\sinh(\sqrt{1-m}\,y)}$ $(k > 0, 0 < m < 1)$, which also describes flow of the sink type in a rotating fluid. Here each trajectory has a pair of asymptotes parallel to one of the lines $y = \pm\sqrt{\frac{m}{1-m}}\,x$. These lines are streamlines on which the vorticity is zero and changes sign upon passing through them. This flow can also be called four-current sink flow. The qualitative pictures of the streamlines for the latter two solutions are shown in Fig. 3. Interest in solutions of this kind was initiated with the work of Zhukovskii [10].

2. Using complex variables, the equation for the stream function (1.1) with right hand side given by $\omega = 4 \sin \psi$ can be written in the form

$$\psi_{z\bar{z}} = \sin\psi,\ z = x + iy,\ \bar{z} = x - iy. \tag{2.1}$$

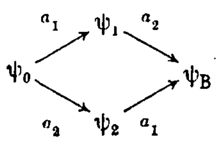
Using the obvious analogy with the equation $\chi_{tx} = \sin\chi$ [5], we immediately write down the Beklund transformation [6]

$$\frac{1}{2}(\psi_z + \varphi_z) = \frac{1}{a}\sin\left(\frac{\psi-\varphi}{2}\right),\quad \frac{1}{2}(\psi_{\bar{z}} - \varphi_{\bar{z}}) = a\sin\left(\frac{\psi+\varphi}{2}\right), \tag{2.2}$$

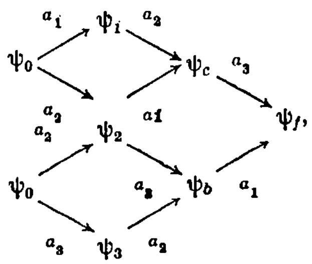
where the parameter a is considered a complex number of modulus unity. It is easy to show that if ψ and φ are solutions of the system (2.2), then each of these functions satisfies (2.1). Therefore if we know one solution φ of (2.1) the other solution can be found by integrating the system (2.2). It is remarkable that this can be done by solving an ordinary differential equation of the first order. Repetition of this procedure leads to a new solution of (2.1), and so on. The repeated integration can be avoided by using the Bianchi permutability theorem [6]. Let $\varphi \equiv \psi_0$ be a known solution of (2.1) and let $\psi_i (i = 1, 2)$ be solutions obtained by integrating the system (2.2) with $a = a_i (i = 1, 2)$. Then the Bianchi theorem can be used to find a new solution ψ_B directly from the equation

$$\tan\frac{\psi_B - \psi_0}{4} = \frac{a_1 + a_2}{a_1 - a_2}\tan\frac{\psi_1 - \psi_2}{4}, \tag{2.3}$$

which is a nonlinear superposition principle for the solutions of (2.1). This process can be illustrated by means of a diagram



The identity (2.3) can be used to construct "N-soliton" solutions. For example, the "three-soliton" solution corresponds to the diagram



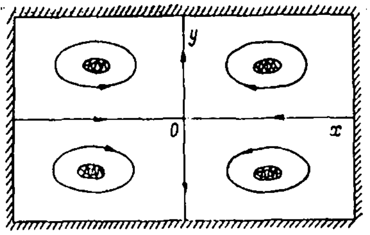


Fig. 4

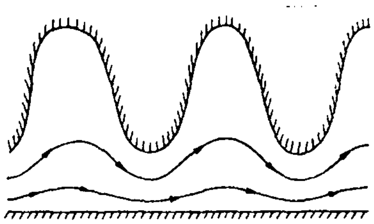


Fig. 5

where $\psi_0 = 0$ and from (2.2) the functions ψ_l (l = 1, 2, 3) have the form

$$\psi_l = 4\arctan\exp(\bar{a}_l z + a_l \bar{z}),\quad a_l = \exp(i\alpha_l),\quad \alpha_l \in R. \tag{2.4}$$

The functions ψ_c and ψ_b are found using the nonlinear superposition principle (2.3)

$$\psi_c = 4\arctan\left(\frac{a_1 + a_2}{a_1 - a_2}\,\frac{\sinh\left(\frac{\sigma_1 - \sigma_2}{2}\right)}{\cosh\left(\frac{\sigma_1 + \sigma_2}{2}\right)}\right),\quad \psi_b = 4\arctan\left(\frac{a_2 + a_3}{a_2 - a_3}\,\frac{\sinh\left(\frac{\sigma_2 - \sigma_3}{2}\right)}{\cosh\left(\frac{\sigma_2 + \sigma_3}{2}\right)}\right)$$

($\sigma_l = \bar{a}_l z + a_l \bar{z}$). The solution ψ_f is given by the relation $\tan[0.25(\psi_f - \psi_2)] = \frac{a_1 + a_3}{a_1 - a_3}\tan[0.25 \times (\psi_c - \psi_b)]$. The final formula for ψ_f is

$$\psi_f = 4\arctan\left[\varkappa_{31}\frac{S(\sigma_2, \sigma_3) - S(\sigma_1, \sigma_2)}{1 - S(\sigma_2, \sigma_3)\,S(\sigma_1, \sigma_2)}\right] + 4\arctan[\exp(\sigma_2)].$$

Here $S(\sigma_i, \sigma_j) = \varkappa_{ij}\dfrac{\sinh[0{,}5(\sigma_i - \sigma_j)]}{\cosh[0{,}5(\sigma_i + \sigma_j)]}$; $\varkappa_{ij} = \cot\left(\dfrac{\alpha_i - \alpha_j}{2}\right)$; σ_i and α_i have been defined above. The functions ψ_i (i = 1, 2, 3) are real, while ψ_c and ψ_b are purely imaginary. A purely imaginary solution of (2.1) is a real solution of the equation $\psi_{\bar{z}z} = \sinh\psi$, which admits a Beklund transformation of the type (2.2) with the trigonometric sines on the right hand side replaced by hyperbolic sines. The trigonometric tangents in (2.3) are then replaced by hyperbolic tangents.

When $a_1 \to a_2$ the solution generating formula (2.3) becomes

$$\tan\frac{\psi_B - \psi_0}{4} = \frac{a}{2}\frac{d}{da}\psi_1. \tag{2.5}$$

If we take $\psi_0 = 0$, then ψ_1 is given by (2.4) and $\psi_B = 4\arctan\left(\dfrac{a\bar{z} - \bar{a}z}{\operatorname{ch}\sigma}\right)$ ($\sigma = \bar{a}z + a\bar{z}$). Repeated application of the superposition principle (2.5) gives the following new solution of (2.1)

$$\psi = 4\arctan(\exp\sigma) + 4\arctan\left[2\,\frac{(\bar{a}z + a\bar{z})\cosh\sigma - (a\bar{z} - \bar{a}z)^2\sinh\sigma}{\cosh^2\sigma + (a\bar{z} - \bar{a}z)^2}\right].$$

When a = 1 this solution takes the form

$$\psi = 4\arctan(\exp 2x) + 4\arctan\left(4\,\frac{x\cosh 2x - 2y^2\sinh 2x}{\cosh^2 2x - 4y^2}\right).$$

It is not difficult to find an explicit expression for the streamlines in this case

$$y^2 = \frac{\cosh 2x}{4}\,\frac{\cosh 2x\,(c - \exp 2x) - 4x\,(1 + \exp 2x)}{c\exp(4x) - \exp(-2x)},$$

where the constant c is intrinsic to each trajectory. Without presenting the detailed proofs, we assert that the velocity is bounded on the entire plane of flow except for two points (0, 1/2) and (0, —1/2) at which it has singularities; all of the trajectories enter or leave from these points; the streamline field is symmetric with respect to the axis x = 0. This describes the motion of a rotating fluid in the presence of a source and a sink. Unlike the potential flow for a dipole, in this case there are streamlines leaving the source and going out to infinity (and coming in from infinity and ending on the sink) in addition to the usual trajectories joining the source and sink. We note that the "N-soliton" solutions

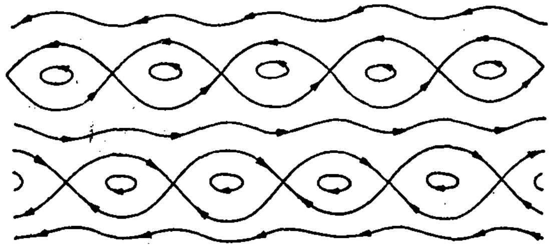


Fig. 6

described above are also not smooth on the entire plane of flow, but have source or sink singularities.

3. We assume that $\omega(\psi) = \sinh\psi$. Then we look for a solution of (1.1) in the form $\psi = 4\,\mathrm{arth}(f(x)g(y))$, where the functions f and g satisfy the equations

$$f'^2 = nf^4 + mf^2 + k,\quad g'^2 = -kg^4 + (1-m)g^2 - n,\quad n, m, k \in R. \tag{3.1}$$

Then

$$u = \frac{4fg'}{1-f^2g^2},\quad v = -\frac{4f'g}{1-f^2g^2},\quad \omega = 4fg\frac{1+f^2g^2}{(1-f^2g^2)^2}. \tag{3.2}$$

If we set n equal to zero in (3.1), we obtain the solution [6]: $g = \sqrt{\frac{1-m}{k}}\frac{1}{\cosh(\sqrt{1-m}\,y)}$, $f = \sqrt{\frac{k}{-m}}\sin(\sqrt{-m}\,x)$ $(k > 0,\ m < 0)$. The straight lines $x = \pi L/\sqrt{-m}$ (L is any integer) are streamlines. Because of the periodicity of f and the fact that it is an odd function, it is sufficient to know the streamlines inside the strip $M = \{0 \leqslant x \leqslant \pi/\sqrt{-m}\}$. Solving the equation for the trajectories $\cosh(\sqrt{1-m}y) = c\sin(\sqrt{m}x)$ for y, $(c > 1)$, it is not difficult to show that all streamlines lying inside M must be closed and bounded. We note that the trajectories in M are symmetric with respect to the straight lines $y = 0$, $x = \pi/2\sqrt{-m}$. It follows at once from (3.2) that the velocity and vorticity go to zero as $|y| \to \infty$ and to infinity as we approach the line $\sqrt{\frac{1-m}{-m}}\frac{\sin(\sqrt{-m}\,x)}{\cosh(\sqrt{1-m}\,y)} = 1$. Taking any two streamlines belonging to the set $\{(x, y) \in M : 0 < f(x)g(y) < 1\}$ as solid boundaries, we obtain the flow field between two cylinders. According to the Arnol'd theorem [11], this flow field will be stable to two-dimensional perturbations since there exist two numbers A_1 and A_2 satisfying the inequality $A_1 \geqslant \omega'(\psi) \equiv \cosh\psi \geqslant A_2 > 0$.

Suppose the parameters k, n, m obey the inequalities $k > 0$, $n < 0$, $0 < m < 1$. Then both of the equations of (3.1) can be reduced by scale transformations to the form

$$h'^2 = (1-h^2)(p_1^2 + p_2^2h^2),\quad p_i \in R,\quad p_1^2 + p_2^2 = 1.$$

This equation is satisfied by the Jacobian cosine elliptic function [7, 8]. Hence the functions f and g can be written in terms of the elliptic cosine function, have periods t_1 and t_2, and have two zeroes per period. In view of the translational invariance of (3.1), we can put $f(0) = g(0) = 0$. Therefore the straight lines $x = t_1L/2$, $y = t_2L/2$ (L is an integer) are streamlines. The streamlines can be obtained over the entire plane of flow knowing only the behavior of the trajectories inside the rectangle $P = \{0 \leqslant x \leqslant t_1/2,\ 0 \leqslant y \leqslant t_2/2\}$. Following the reasoning of Sec. 1 (case *a*), we conclude that all trajectories lying inside P must be closed and bounded. However the stream function, velocity, and vorticity all go to infinity as we approach the line $fg = 1$. The existence of such a line follows from the fact that f varies between $-\sqrt{\frac{-m-\sqrt{m^2-4kn}}{2n}}$ and $\sqrt{\frac{-m-\sqrt{m^2-4kn}}{2n}}$, while the function g varies from $-\sqrt{\frac{1-m+\sqrt{(1-m)^2-4kn}}{2k}}$ to $\sqrt{\frac{1-m+\sqrt{(1-m)^2-4kn}}{2k}}$. Indeed, according to (3.1), the extremum values of the functions can be found from the solution of the biquadratic equations $nf^4 + mf^2 + k = 0$, $kg^4 + (-1+m)g^2 + n = 0$. It follows from the above discussion that it is necessary to consider a set of the type $\Pi_c = \{(x, y) \in R^2: 0 \leqslant x \leqslant t_1, 0 \leqslant y \leqslant t_2, f^2g^2 \leqslant c < 1\}$ on the plane of flow. If we take the boundary Π_c to be a solid wall, then we obtain stable [11] vortex flow inside a rectangular cylinder in which there are four symmetric cylinders. The streamlines are shown in Fig. 4, where the vorticity changes sign when passing through a symmetry axis of the rectangle.

Periodic motion between a flat bottom and a cover can be obtained for the following conditions on the parameters of (3.1): $m < 0$, $n > 0$, $m^2 = 4kn$. With the help of scale transformations, (3.1) for g reduces to the form $g'^2 = (1 - g^2)(g^2 - p^2)$ $(p \in R)$. Hence g can be expressed in terms of the elliptic delta function dny [7, 8], which is periodic and nowhere zero. The function f is given by $f = \sqrt{\frac{-m}{2n}} \tanh\left(\sqrt{\frac{-m}{2}}\, x\right)$. The equation for the streamlines $\sqrt{\frac{-m}{2n}} \tanh\left(\sqrt{\frac{-m}{2}}\, x\right) g(y) = c$ when solved for x has the form $x(y) = \sqrt{\frac{2}{-m}}\, \mathrm{arth}\left(\frac{c\sqrt{\frac{2n}{m}}}{g(y)}\right)$. Equation (3.1) can be used to find the region of allowed values of the function g: $\sqrt{\frac{1 - m - \sqrt{1 - 2m}}{2k}} \leqslant g(y) \leqslant \sqrt{\frac{1 - m + \sqrt{1 - 2m}}{2k}}$. Suppose we have the inequality $\frac{c}{\min g}\sqrt{\frac{2n}{-m}} < 1$ or its equivalent $c < \sqrt{\frac{1 - m - \sqrt{1 - 2m}}{-m}}$. Then x(y) will be a smooth periodic function. If we put $c = \sqrt{\frac{1 - m - \sqrt{1 - 2m}}{-m}}$, then the function x(y) will be determined everywhere except at the points at which g(y) reaches minimum values. Taking the straight line $x = 0$ and any streamline belonging to the set $\left\{(x, y) \in R^2: f(x)\, g(y) < \sqrt{\frac{1 - m - \sqrt{1 - 2m}}{-m}}\right\}$, as solid walls, we obtain periodic motion in y. The streamlines are shown in Fig. 5.

Equations of the form (1.1) arise in a wide range of physical problems: problems of plasma physics [12], statistical mechanics [13, 14], and steady heat conduction. For example, two of the solutions described above were discussed in a study of solitons in the hydrodynamic model of a cold plasma [13].

4. Suppose the vorticity in (1.1) has the form $\omega = \beta \exp(-2\psi) - \exp \psi$. Then we can look for a solution of (1.1) in the form $\psi = \ln(f(x) + g(y))$, where the functions f and g satisfy the ordinary differential equations

$$f'^2 = -2f^3 + a_2f^2 + a_1f + a_0,\quad g'^2 = -2g^3 - a_2g^2 + a_1g - a_0 - \beta. \tag{4.1}$$

We will assume that $a_1 = 0$ and $\beta > 0$, $a_2 < 0$, $a_0 > 0$. The components of the velocity and the vorticity are given by

$$u = g'/(f + g),\quad v = -f'/(f + g),\quad \omega = \beta/(f + g)^2 - f - g. \tag{4.2}$$

In order that f and g be periodic functions, each of the cubic equations $-2f^3 + a_2f^2 + a_0 = 0$, $-2g^3 - a_2g^2 - a_0 - \beta = 0$ must have three unequal real roots. If $f_1 < f_2 < f_3$ are the roots of the first equation then $-f_1$, $-f_2$, $-f_3$ are the roots of the equation $-2g^3 - a_2g^2 - a_0 = 0$ and so from the above assumptions we have $f_1 < 0$, $f_2 < 0$, $f_3 > 0$. Hence for small β the equation $-2g^3 - a_2g^2 - a_0 - \beta = 0$ will also have three roots $g_1 > g_2 > g_3$ $(g_2 > 0,\ g_3 < 0)$. Therefore there exist numbers a_0, a_2, β such that the functions f and g satisfying (4.1) can be chosen to be periodic. Since $\beta > 0$, the sum $f_2 + g_2$ is positive. As in the preceding paragraphs, we can find the region of values of the functions f and g: $f_2 \leqslant f(x) \leqslant f_3$, $g_2 \leqslant g(y) \leqslant g_1$. Hence the mapping $\ln(f(x) + g(y))$ is defined correctly.

Let t_1 and t_2 be the periods of the functions f and g, respectively. Because of the translational invariance of (4.1), we can assume that f and g have maximum values at zero. Then f will reach maximum values at the points t_1L and minimum values at the points $t_1(2L + 1)/2$ (L is any integer). Similarly the function g reaches maximum values at the points t_2L and minimum values at the points $t_2(2L + 1)/2$. The points of the plane $R^2(x, y)$ where the functions f and g both have a maximum or a minimum are singular points of the differential equations for the streamlines $dx/u = dy/v$, which, in view of (4.2), are equivalent to

$$dx/g' = dy/-f'. \tag{4.3}$$

The points of the plane where f has a maximum and g has a minimum (or vice versa) are saddle points for (4.3).

The trajectories are the isolines of the function F(x, y) = f(x) + g(y). If the gradient of F vanishes at a point, this point is called a critical point, and the corresponding value of F is a critical value. Only those points of the plane at which the derivatives of the functions f and g both vanish are critical points for F. Hence in general there exist four critical values: $c_1 = \max_{x\in R} f(x) + \max_{y\in R} g(y)$, $c_2 = \max_{y\in R} g(y) + \min_{x\in R} f(x)$, $c_3 = \min_{y\in R} g(y) + \max_{x\in R} f(x)$, $c_4 = \min_{y\in R} g(y) + \min_{x\in R} f(x)$. From the regular interval theorem [9], the isolines $F(x, y) = s_1$, $F(x, y) = s_3$, where $c_1 < s_1 < c_2$, $c_3 < s_3 < c_4$, are diffeomorphic circles, since the critical points of F corresponding to the critical values c_1 and c_4 are central singular points for (4.3). The isolines corresponding to the critical values c_2 and c_3 pass through the saddle points of (4.3). They are separatrices of (4.3) and pass from saddle point to saddle point. Suppose $c_2 > c_3$. Then the separatrices join the saddle points lying on the straight lines $y = t_2L/2$ (L is an integer). The qualitative form of the corresponding streamlines is shown in Fig. 6, in which vortex chains are arranged in a staggered order. Because $c_2 \neq c_3$, there exist streamlines in the plane of flow going between the chains. If $c_2 = c_3$ the boundaries of neighboring chains become common boundaries. The qualitative form of the streamlines in the case $c_3 > c_2$ can be obtained by turning Fig. 6 by 90°. Note the similarity between the solutions discussed in this section and the periodic secondary flows described in [15].

The author thanks V. A. Vladirimov for detailed discussions of the results.

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